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PRODUCT OF GENERATOR MATRICES OF KNOWN SEQUENCES AND SALIENT FEATURES

CHANDRAKANT B. PATEL[1] & PRADEEP J. JHA[2]

[1]Research Scholar, Calorx Teachers' University, Ahmadabad, Gujarat, India

[2]Research Guide, Calorx Teachers' University, Ahmadabad, Gujarat, India

ABSTRACT

In this article we discuss some important salient features of different matrices obtained by the result of product of generator matrices of some known sequences. These features, as the study indicates, inherit some properties, like recurrence relation, and irrational part of Eigen values to that of original sequences considered for the purpose.

We have also seen that, in some cases, these properties parallel to corresponding terms in the generalized form of the product matrix and hence giving rise to new sequences.

KEYWORDS: Pell Sequence, Fibonacci sequence, Jha Sequence, Operator Matrix & Eigen Values

1. INTRODUCTION TO SOME KNOWN SEQUENCES

As we want to treat with the generator matrices of same known sequence; it is Important to introduce some known infinite sequence of positive integers. We consider three sequences in order and highlight some salient features that will help us carry further algebraic operations.

1.1. Jha Sequence

In fact there are different ways to introduce this 'Jha' sequence of non-negative integers. It is a sequence of terms in which each term added to its next term results in to the sum of two legs of right triangle of consecutive right triangle of Fermat family. Some the right triangles of Fermat family are shown below and the terms of the Jha sequence will clarify its origin. It has been, on extensive work on Jha sequence, found that it is very useful in finding the general form of the sum of infinite terms of Pell sequence and general ratio of consecutive terms to many sequences.

Table 1: Fermat Triangles

Sr No.	Legs		Hypotenuse
	a	b	h
1	0	1	1
2	3	4	5
3	20	21	29
4	119	120	169
5	696	697	985

Some terms of 'Jha' sequence (J_n) are

Jn: 0, 1, 6, 35, 204 ... **(1)**

t_1= 0, t_2= 1, t_3= 6, t_4= 35, t_5 = 204...

Now

- $\boldsymbol{t_1} + \boldsymbol{t_2} = \mathbf{0 + 1 = 1} = \boldsymbol{a_1} + \boldsymbol{b_1} = \mathbf{0 + 1 = 1}$
- $\boldsymbol{t_2} + t_3 = \mathbf{1 + 6 = 7} = \boldsymbol{a_2} + \boldsymbol{b_2} = \mathbf{3 + 4 = 7}$
- $\boldsymbol{t_3} + \boldsymbol{t_4} = \mathbf{6 + 35 = 41} = \boldsymbol{a_3} + \boldsymbol{b_3} = \mathbf{20 + 21 = 41}$ etc.

The sequence defined by **(1)** has a recurrence relation. It is given as

$\boldsymbol{t_{n+2}} = \mathbf{6}\, \boldsymbol{t_{n+1}} - \boldsymbol{t_n}$ with $t_1 = 0$, $t_2 = 1$ for all n ∈ N **(2)**

We can derive the general formula for the nth term; it is as follows

$\mathbf{J_n} = \frac{\sqrt{2}}{8}\left[\left(3+2\sqrt{2}\right)^{n-1} - \left(3-2\sqrt{2}\right)^{n-1}\right]$ for n∈ N. **(3)**

The generator matrix of the sequence is denoted as J; $\mathbf{J} = \begin{pmatrix} 6 & -1 \\ 1 & 0 \end{pmatrix}$ **(4)**

1.2. Pell Sequence

An irrational root of a positive integer is always an approximation which by some technique is claimed to be better than the one found using some iterative formula.

One of such equation used for searching always better approximation

For $\sqrt{2}$ is given by the equation $X^2 - 2Y^2 = \pm 1$. **(5)**

Where X and Y are positive integers

Integers satisfying the equation **(5)** are such that their ratio $\frac{x}{y}$ gives approximation for $\sqrt{2}$ and this ratio is improved as X and Y simultaneously increases satisfying equation **(5)**

The sequence of ratio is $\{\frac{x}{y} = \frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \ldots\}$

The terms appearing in the denominator i.e. 1, 2, 5, 12,... is called Pell sequence. To pace

With certain mathematical structure we include '0' and reformat it as

$\mathbf{P_n} = \mathbf{0, 1, 2, 5, 12, 29 \ldots}$ **(6)**

Pell sequence obeys advancing rule; called the recurrence relation given as follows.

$\mathbf{t_{n+2} = 2t_{n+1} + t_n}$ ∀ n∈ N ; with $t_1=0$, and $t_2=1$, **(7)**

the general term is given as

$\mathbf{P_n} = \frac{\sqrt{2}}{4}\left[\left(1+\sqrt{2}\right)^{n-1} - \left(1-\sqrt{2}\right)^{n-1}\right]$ for all n ∈ N **(8)**

Also the generator matrix is given denoted as P is; $\mathbf{P} = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$ **(9)**

1.3. Fibonacci Sequence

It is a well known and probably the oldest one which is used in many mathematical applications. Its advancing rule and other features are very close to rules of nature

The infinite sequence, Fibonacci sequence is as follows.

$\{F_n$=1, 1, 2, 3, 5, 8, 13, 21......} **(10)**

[each term after the second term appears as the sum of two previous terms.]

The recurrence relation is

$F_{n+2} = F_{n+1} + F_n$, and F1 = 1, F_2 = 1for n ∈ N **(11)**

The general term of the sequence is $\mathbf{F_n} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\right]$ for all n ∈ N **(12)**

Its generator matrix denoted as F is; F= $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ **(13)**

2. ALGEBRAIC OPERATION ON THE GENERATOR MATRICES

In this section, we consider the generator matrices of different sequences mentioned above. We consider the matrix obtained by the product of two generator matrices of two sequences. Then we find the set of Eigen values of the product matrix and that of all the different matrices obtained by taking different exponents of the basic matrix. We observe that Eigen values of the successive matrices exhibit traits of original recurrence relation.

2.1. Product of Generator Matrices of Jha Sequence and Pell Sequence

As said earlier, we consider from the above unit, the generator matrix J = $\begin{pmatrix} \mathbf{6} & \mathbf{-1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}$ of Jha sequence and P = $\begin{pmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}$ of Pell sequence and find their product JP as follows

The product of J and P matrices denoted as **[JP]**

$$[JP] = \begin{pmatrix} \mathbf{6} & \mathbf{-1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}\begin{pmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{11} & \mathbf{6} \\ \mathbf{2} & \mathbf{1} \end{pmatrix} \quad \textbf{(14)}$$

We compute different powers of the matrix PJ and enlist the result as follow.

$$[JP]^2 = \begin{pmatrix} \mathbf{133} & \mathbf{72} \\ \mathbf{24} & \mathbf{13} \end{pmatrix} \qquad [\boldsymbol{JP}]^3 = \begin{pmatrix} \mathbf{1607} & \mathbf{870} \\ \mathbf{290} & \mathbf{157} \end{pmatrix} \qquad [\boldsymbol{JP}]^4 = \begin{pmatrix} \mathbf{19417} & \mathbf{10512} \\ \mathbf{3504} & \mathbf{1897} \end{pmatrix}$$

and hence in general we write its n^{th} product as

$$[\boldsymbol{JP}]^n = \begin{pmatrix} \mathbf{11}\sum_{i=1}^{n}(-1)^{n+i}t_i + \sum_{i=1}^{n-1}(-1)^{n+i}t_i & \mathbf{6}\sum_{i=1}^{n}(-1)^{n+i}t_i \\ \mathbf{2}\sum_{i=1}^{n}(-1)^{n+i}t_i & \boldsymbol{t_n} \end{pmatrix}$$

Where t_n is corresponding positional term in the matrix $(JP)^n$ for n = 1,2,3...

Now we find Eigen values of these matrices and study their features.

Sr. No.	Matrix	Eigen values
1	JP = $\begin{pmatrix} \mathbf{11} & \mathbf{6} \\ \mathbf{2} & \mathbf{1} \end{pmatrix}$	$(\mathbf{6} \pm \sqrt{\mathbf{37}})$
2	$[JP]^2 = \begin{pmatrix} \mathbf{133} & \mathbf{72} \\ \mathbf{24} & \mathbf{13} \end{pmatrix}$	$(\mathbf{73} \pm \mathbf{12}\sqrt{\mathbf{37}})$
3	$[JP]^3 = \begin{pmatrix} \mathbf{1607} & \mathbf{870} \\ \mathbf{290} & \mathbf{157} \end{pmatrix}$	$(\mathbf{882} \pm \mathbf{145}\sqrt{\mathbf{37}})$
4	$[JP]^4 = \begin{pmatrix} \mathbf{19417} & \mathbf{10512} \\ \mathbf{3504} & \mathbf{1897} \end{pmatrix}$	$(\mathbf{10657} \pm \mathbf{1752}\sqrt{\mathbf{37}})$

We derive more observation from the matrices derived above. Considering fixed positional terms; we have

t_1=1 from $[\boldsymbol{JP}]$

t_2=13 from$[\boldsymbol{JP}]^2$

t_3=157 from$[Jp]^3$...

We observe that all the entries of the matrix $[\boldsymbol{JP}]^n$ are expressed in terms of the sequence $t_{n.}$

The sequence t_n is expressed as follows.

(t_n) =1, 13, 157, 1897… $n \in N$

There are two dominant features which are as follows.

With t_1=1, t_2=13, the recurrence relation is $\mathbf{t_{n+2} = 12t_{n+1}+t_n}$ $\forall$ n$\in N$ **(15)**

The general term is

$$\mathbf{t_n} = \frac{\mathbf{1}}{\mathbf{2\sqrt{37}}}\{(\mathbf{7}+\sqrt{\mathbf{37}})(\mathbf{6}+\sqrt{\mathbf{37}})^{\mathbf{n-1}} - (\mathbf{7}-\sqrt{\mathbf{37}})(\mathbf{6}-\sqrt{\mathbf{37}})^{\mathbf{n-1}}\} \quad \mathbf{(16)}$$

In order to make further analysis we study the sequence given by the above result.

We have its generator matrix $\begin{pmatrix} \mathbf{12} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}$ =**W** say **(17)**

As clearly observed from the terms of the sequence (t_n): 1, 13, 157, 1897…

with t_1=1, and t_2 = 13

$$\begin{pmatrix} t_3 \\ t_2 \end{pmatrix} = \begin{pmatrix} \mathbf{12} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}\begin{pmatrix} t_2 \\ t_1 \end{pmatrix} = \mathrm{W}\begin{pmatrix} t_2 \\ t_1 \end{pmatrix}$$

$$\begin{pmatrix} t_4 \\ t_3 \end{pmatrix} = \begin{pmatrix} \mathbf{12} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}^2\begin{pmatrix} t_2 \\ t_1 \end{pmatrix} = \begin{pmatrix} \mathbf{145} & \mathbf{12} \\ \mathbf{12} & \mathbf{1} \end{pmatrix} = \mathrm{W}^2\begin{pmatrix} t_2 \\ t_1 \end{pmatrix}$$

Similarly

$$\begin{pmatrix} t_5 \\ t_4 \end{pmatrix} = \begin{pmatrix} \mathbf{12} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}\begin{pmatrix} t_2 \\ t_1 \end{pmatrix} = \begin{pmatrix} \mathbf{1752} & \mathbf{145} \\ \mathbf{145} & \mathbf{12} \end{pmatrix} = \mathrm{W}^3\begin{pmatrix} t_2 \\ t_1 \end{pmatrix}$$

Continuing in the same pattern, we have

$$\begin{pmatrix} t_n \\ t_{n-1} \end{pmatrix} = \begin{pmatrix} \mathbf{12} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}^{n-2} \begin{pmatrix} t_2 \\ t_1 \end{pmatrix} \ \ or \ \ \begin{pmatrix} t_{n+2} \\ t_{n+1} \end{pmatrix} = \begin{pmatrix} \mathbf{12} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}^{n} \begin{pmatrix} t_2 \\ t_1 \end{pmatrix} \ \ for\ n \in N$$

$$\therefore [\boldsymbol{W}]^n = \begin{pmatrix} \mathbf{12} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}^{n} = \begin{pmatrix} \sum_{i=1}^{n} 12(-1)^{n+i} t_i + \sum_{i=0}^{n-1} (-1)^{(n-1)+i} t_i & \sum_{i=1}^{n} (-1)^{n+i} t_i \\ \sum_{i=1}^{n} (-1)^{n+i} t_i & \sum_{i=0}^{n-1} (-1)^{i+(n-1)} t_i \end{pmatrix}_{2\times 2}$$

Now we enlist the result of Eigen values to the different exponents of the matrix W^n.

Matrix	**Eigen Values**
W	$(EW)^1 = (\mathbf{6} \pm \sqrt{\mathbf{37}})$
W^2	$(EW)^2 = (\mathbf{73} \pm \mathbf{12}\sqrt{\mathbf{37}})$
W^3	$(EW)^3 = (\mathbf{882} \pm \mathbf{145}\sqrt{\mathbf{37}})$|
W^n	$(EW)^n = [(6\sum_{i=1}^{n}(-1)^{n+i}t_i + \sum_{i=1}^{n-1}(-1)^{n+i}t_i) \pm (\sum_{i=1}^{n}(-1)^{n+i}t_i)\sqrt{37}]$

We make the following observation from the pattern of Eigen values.

- $\sqrt{37}$ Remains constant and appears on the eigen values of all the matrices. Considering a generalized form of the generator matrix W^n ; we put its Eigen value as $\alpha_n+\beta_n\sqrt{37}$

  for different values of n$\in N$.

  Now, we note an important point that $\lim_{n\to\infty}\frac{(\alpha n)}{(\beta n)}$ is a rational approximation of $\sqrt{37}$ the approximating sequence is $\frac{6}{1}, \frac{73}{12}, \frac{882}{145}, \frac{10657}{1752}, \ldots$

- From the two parts (rational and irrational) of Eigen values, we have two different pattern of sequence 6, 73,882, 10657… and 1, 12, 145, 1752…

  The important point is that the terms of these two sequences follow the same recurrence relation.

2.2. Product of Generator Matrices of Fibonacci sequence and Pell Sequence

As said earlier, we consider from the above unit, the generator matrix $F = \begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}$ of Fibonacci sequence and P = $\begin{pmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}$ of Pell sequence and find their product **denoted as [FP]**

$$[FP] = \begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}\begin{pmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{3} & \mathbf{2} \\ \mathbf{1} & \mathbf{1} \end{pmatrix} \ \boldsymbol{and} \ [PF] = \begin{pmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}\begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{3} & \mathbf{1} \\ \mathbf{2} & \mathbf{1} \end{pmatrix} \quad \mathbf{(18)}$$

$$\therefore [\boldsymbol{PF}] = [FP]^T$$

We compute different powers of the matrix PJ and enlist the result as follow.

$$[\boldsymbol{FP}]^2 = \begin{pmatrix} \mathbf{11} & \mathbf{4} \\ \mathbf{8} & \mathbf{3} \end{pmatrix} \qquad [\boldsymbol{FP}]^3 = \begin{pmatrix} \mathbf{41} & \mathbf{15} \\ \mathbf{30} & \mathbf{11} \end{pmatrix} \qquad [\boldsymbol{FP}]^4 = \begin{pmatrix} \mathbf{153} & \mathbf{56} \\ \mathbf{112} & \mathbf{41} \end{pmatrix}$$

and hence in general we write its n^{th} product as

$$[\boldsymbol{FP}]^n = \begin{pmatrix} \mathbf{3}\boldsymbol{t_n} + \mathbf{2}\boldsymbol{t_{n-1}} & \boldsymbol{t_n} + \boldsymbol{t_{n-1}} \\ \mathbf{2}(\boldsymbol{t_n} + \boldsymbol{t_{n-1}}) & \boldsymbol{t_n} \end{pmatrix}$$

Where t_n is corresponding positional term in the matrix $(FP)^n$ for n = 1,2,3...

Hence as it can be observed

Sr. No.	Matrix	Eigen values
1	$\mathbf{FP} = \begin{pmatrix} \mathbf{3} & \mathbf{2} \\ \mathbf{1} & \mathbf{1} \end{pmatrix}$	$(\mathbf{2} \pm \mathbf{1}\sqrt{\mathbf{3}})$
2	$[FP]^2 = \begin{pmatrix} \mathbf{11} & \mathbf{4} \\ \mathbf{8} & \mathbf{3} \end{pmatrix}$	$(\mathbf{7} \pm \mathbf{4}\sqrt{\mathbf{3}})$
3	$[FP]^3 = \begin{pmatrix} \mathbf{41} & \mathbf{15} \\ \mathbf{30} & \mathbf{11} \end{pmatrix}$	$(\mathbf{26} \pm \mathbf{15}\sqrt{\mathbf{3}})$
4	$[FP]^4 = \begin{pmatrix} \mathbf{153} & \mathbf{56} \\ \mathbf{112} & \mathbf{41} \end{pmatrix}$	$(\mathbf{97} \pm \mathbf{56}\sqrt{\mathbf{3}})$

We derive more observation from the matrices derived above. Considering fixed positional terms; we have

t_1=1 from $[\boldsymbol{FP}]$

t_2=3 from$[\boldsymbol{FP}]^2$

t_3=11 from$[FP]^3$...

We observe that all the entries of the matrix $[\boldsymbol{FP}]^n$ are expressed in terms of the sequence $t_{n.}$

The sequence t_n is expressed as follows.

(t_n) = 1, 3, 11, 41,153 ... **(19)**

For which two dominant features are as follows.

With $t_1=1$, $t_2=3$ and $\mathbf{t_{n+2} = 4\, t_{n+1} - t_n}$ $\forall$ n$\in N$ **(20)**

Which is its recurrence relation and its general term is

$$\mathbf{t_n} = \frac{\mathbf{1}}{\mathbf{2\sqrt{3}}}\{(\mathbf{1}+\sqrt{\mathbf{3}})(\mathbf{2}+\sqrt{\mathbf{3}})^{\mathbf{n-1}} - (\mathbf{1}-\sqrt{\mathbf{3}})(\mathbf{2}-\sqrt{\mathbf{3}})^{\mathbf{n-1}}\} \quad \textbf{(21)}$$

In order to make further analysis we study the sequence given by the result (19) and let it be a sequence S1, we have its generator matrix $\begin{pmatrix} \mathbf{4} & \mathbf{-1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}$ =**X** say

$$\begin{pmatrix} t_3 \\ t_2 \end{pmatrix} = \begin{pmatrix} \mathbf{4} & \mathbf{-1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}\begin{pmatrix} t_2 \\ t_1 \end{pmatrix} = \mathrm{X}\begin{pmatrix} t_2 \\ t_1 \end{pmatrix}$$

$$\begin{pmatrix} t_4 \\ t_3 \end{pmatrix} = \begin{pmatrix} \mathbf{4} & \mathbf{-1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}\begin{pmatrix} t_2 \\ t_1 \end{pmatrix} = \begin{pmatrix} \mathbf{15} & \mathbf{-4} \\ \mathbf{4} & \mathbf{-1} \end{pmatrix} = \mathrm{X}^2\begin{pmatrix} t_2 \\ t_1 \end{pmatrix}$$

Similarly

$$\begin{pmatrix} t_5 \\ t_4 \end{pmatrix} = \begin{pmatrix} \mathbf{4} & \mathbf{-1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}\begin{pmatrix} t_2 \\ t_1 \end{pmatrix} = \begin{pmatrix} \mathbf{56} & \mathbf{-15} \\ \mathbf{15} & \mathbf{-4} \end{pmatrix} = \mathrm{X}^3\begin{pmatrix} t_2 \\ t_1 \end{pmatrix}$$

Continuing in the same pattern, we have

$$\begin{pmatrix} t_n \\ t_{n-1} \end{pmatrix} = \begin{pmatrix} \mathbf{4} & \mathbf{-1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}^{\mathrm{n-2}}\begin{pmatrix} t_2 \\ t_1 \end{pmatrix} \quad or \quad \begin{pmatrix} t_{n+2} \\ t_{n+1} \end{pmatrix} = \begin{pmatrix} \mathbf{4} & \mathbf{-1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}^{\mathrm{n}}\begin{pmatrix} t_2 \\ t_1 \end{pmatrix} \quad for\ n \in N$$

$$\therefore [\boldsymbol{X}]^n = \begin{pmatrix} \mathbf{4} & \mathbf{-1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}^n = \begin{pmatrix} \sum_{i=1}^{n+1} t_i & -\sum_{i=1}^{n} t_i \\ \sum_{i=1}^{n} t_i & -\sum_{i=0}^{n-1} t_i \end{pmatrix}_{2\times 2}$$

Now we enlist the result of Eigen values to the different exponents of the matrix X^n.

Matrix	**Eigen values**
X	$(EX)^1 = (\mathbf{2} \pm \sqrt{\mathbf{3}})$
X^2	$(EX)^2 = (\mathbf{7} \pm \mathbf{4}\sqrt{\mathbf{3}})$
X^3	$(EX)^3 = (\mathbf{26} \pm \mathbf{15}\sqrt{\mathbf{3}})$
X^n	$(EX)^n = [(2t_n + \sum_{i=0}^{n-1} t_i) \pm (\sum_{i=1}^{n} t_i)\sqrt{3}]$

There are some observations.

- $\sqrt{3}$ remains constant and appears on the Eigen values of all the matrices. Considering a generalized form of the generator matrixW^n; we put its Eigen value as $\propto_n + \beta_n\sqrt{3}$ for different values of n $\in N$.

Now, we note an important point that $\lim_{n\to\infty}\frac{(\alpha n)}{(\beta n)}$ is a rational approximation of $\sqrt{3}$ the approximating sequence is $\frac{2}{1}, \frac{7}{4}, \frac{26}{15}, \frac{97}{56}, \ldots$

- From the two parts (rational and irrational) of Eigen values, we have two different pattern of sequence 2, 7, 26… and 1, 4, 15…

The important point is that the terms of these two sequences follow the same recurrence relation.

2.3. Product of Generator Matrices of Jha Sequence and Fibonacci sequence

As said earlier, we consider from the above unit, the generator matrix $J = \begin{pmatrix} \mathbf{6} & \mathbf{-1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}$ of Jha sequence and F = $\begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}$ of Fibonacci sequence and find their product JF as follows

The product of J and F matrices denoted as [JF]

$$[JF] = \begin{pmatrix} \mathbf{6} & \mathbf{-1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}\begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{5} & \mathbf{6} \\ \mathbf{1} & \mathbf{1} \end{pmatrix} \quad \textbf{(22)}$$

We compute different powers of the matrix PJ and enlist the result as follow.

$$[\boldsymbol{JF}]^2 = \begin{pmatrix} \mathbf{31} & \mathbf{36} \\ \mathbf{6} & \mathbf{7} \end{pmatrix} \qquad [\boldsymbol{JF}]^3 = \begin{pmatrix} \mathbf{191} & \mathbf{222} \\ \mathbf{37} & \mathbf{43} \end{pmatrix} \qquad [\boldsymbol{JF}]^4 = \begin{pmatrix} \mathbf{1177} & \mathbf{1368} \\ \mathbf{228} & \mathbf{265} \end{pmatrix}$$

and hence in general we write its n^{th} product as

$$[\boldsymbol{JF}]^n = \begin{pmatrix} \mathbf{5}\sum_{i=1}^{n}(-1)^{n+i}t_i + \sum_{i=1}^{n-1}(-1)^{n+i}t_i & \mathbf{6}\sum_{i=1}^{n}(-1)^{n+i}t_i \\ \sum_{i=1}^{n}(-1)^{n+i}t_i & \boldsymbol{t_n} \end{pmatrix}$$

Where t_n is corresponding positional term in the matrix $(FJ)^n$ for n = 1,2,3...

Hence as it can be observed

Sr. No.	Matrix	Eigen values
1	$\mathbf{JF} = \begin{pmatrix} \mathbf{5} & \mathbf{6} \\ \mathbf{1} & \mathbf{1} \end{pmatrix}$	$\left(\mathbf{3} \pm \sqrt{\mathbf{10}}\right)$
2	$[JF]^2 = \begin{pmatrix} \mathbf{31} & \mathbf{36} \\ \mathbf{6} & \mathbf{7} \end{pmatrix}$	$\left(\mathbf{19} \pm \mathbf{6}\sqrt{\mathbf{10}}\right)$
3	$[JF]^3 = \begin{pmatrix} \mathbf{191} & \mathbf{222} \\ \mathbf{37} & \mathbf{43} \end{pmatrix}$	$\left(\mathbf{117} \pm \mathbf{37}\sqrt{\mathbf{10}}\right)$
4	$[JF]^4 = \begin{pmatrix} \mathbf{1177} & \mathbf{1368} \\ \mathbf{228} & \mathbf{265} \end{pmatrix}$	$\left(\mathbf{721} \pm \mathbf{228}\sqrt{\mathbf{10}}\right)$

We derive more observation from the matrices derived above. Considering fixed positional terms; we have

$t_1=1$ from $[\boldsymbol{JF}]$

$t_2=7$ from$[\boldsymbol{JF}]^2$

$t_3=43$ from$[JF]^3$...

We observe that all the entries of the matrix $[\boldsymbol{JF}]^n$ are expressed in terms of the sequence t_n.

The sequence t_n is expressed as follows.

$(t_n)_n \in N = 1, 7, 43, 265, 1633,$ **(23)**

For which two dominant features are as follows.

With $t_1=1$, $t_2=7$ and $\mathbf{t_{n+2} = 6\, t_{n+1} + t_n}$ $\forall$ n$\in N$ **(24)**

Which is its recurrence relation and its general term is

$$\mathbf{t_n} = \frac{\mathbf{1}}{\mathbf{2\sqrt{10}}}\{\left(\mathbf{4} + \sqrt{\mathbf{10}}\right)\left(\mathbf{3} + \sqrt{\mathbf{10}}\right)^{\mathbf{n-1}} - \left(\mathbf{4} - \sqrt{\mathbf{10}}\right)\left(\mathbf{3} - \sqrt{\mathbf{10}}\right)^{\mathbf{n-1}}\} \quad \textbf{(25)}$$

The generator matrix of Z-sequence $Z = \begin{pmatrix} \mathbf{6} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}$

$$\begin{pmatrix} t_3 \\ t_2 \end{pmatrix} = \begin{pmatrix} \mathbf{6} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix} \begin{pmatrix} t_2 \\ t_1 \end{pmatrix} = \mathrm{Z} \begin{pmatrix} t_2 \\ t_1 \end{pmatrix}$$

$$\begin{pmatrix} t_4 \\ t_3 \end{pmatrix} = \begin{pmatrix} \mathbf{37} & \mathbf{6} \\ \mathbf{6} & \mathbf{1} \end{pmatrix} \begin{pmatrix} t_2 \\ t_1 \end{pmatrix} = \begin{pmatrix} \mathbf{37} & \mathbf{6} \\ \mathbf{6} & \mathbf{1} \end{pmatrix} = \mathrm{Z}^2 \begin{pmatrix} t_2 \\ t_1 \end{pmatrix}$$

$$\begin{pmatrix} t_5 \\ t_4 \end{pmatrix} = \begin{pmatrix} t_2 \\ t_1 \end{pmatrix} = \begin{pmatrix} \mathbf{228} & \mathbf{37} \\ \mathbf{37} & \mathbf{6} \end{pmatrix} = \mathrm{Z}^3 \begin{pmatrix} t_2 \\ t_1 \end{pmatrix}$$

Continuing in the same pattern, we have

$$\begin{pmatrix} t_n \\ t_{n-1} \end{pmatrix} = \begin{pmatrix} \mathbf{6} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}^{\mathrm{n-2}} \begin{pmatrix} t_2 \\ t_1 \end{pmatrix} \quad or \quad \begin{pmatrix} t_{n+2} \\ t_{n+1} \end{pmatrix} = \begin{pmatrix} \mathbf{6} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}^{\mathrm{n}} \begin{pmatrix} t_2 \\ t_1 \end{pmatrix} \quad for\ n \in N$$

$$\therefore [\boldsymbol{Z}]^n = \begin{pmatrix} \mathbf{6} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}^{\mathrm{n}} = \begin{pmatrix} \sum_{i=1}^{n+1}(-1)^{n+i}t_i & \sum_{i=1}^{n}(-1)^{n+i}t_i \\ \sum_{i=1}^{n}(-1)^{n+i}t_i & \sum_{i=0}^{n-1}(-1)^{(n-1)+i}t_i \end{pmatrix}_{2\times 2}$$

Now we enlist the result of Eigen values to the different exponents of the matrix Z^n.

Matrix	**Eigen values**
Z	$(EZ)^1 = \left(\mathbf{3 \pm \sqrt{10}}\right)$
Z^2	$(EZ)^2 = \left(\mathbf{19 \pm 6\sqrt{10}}\right)$
Z^3	$(EZ)^3 = \left(\mathbf{117 \pm 37\sqrt{10}}\right)$
|	
|	
Z^n	$(EZ)^n = \left[\left(3\sum_{i=1}^{n}(-1)^{n+i}t_i\right) + \sum_{i=0}^{n-1}(-1)^{(n-1)+i}t_i\right) \pm \left(\sum_{i=1}^{n}(-1)^{n+i}t_i\right)\sqrt{10}\right]$

We have some observations,

- $\sqrt{10}$ remains constant and appears on the Eigen values of all the matrices. Considering a generalized form of the generator matrixW^n; we put its Eigen value as $\propto_n + \beta_n\sqrt{10}$ for different values of n $\in N$.

Now, we note an important point that $\lim_{n\to\infty}\frac{(\alpha n)}{(\beta n)}$ is a rational approximation of $\sqrt{10}$ the approximating sequence is $\frac{3}{1}, \frac{19}{6}, \frac{117}{37}, \frac{721}{228}, \ldots$

- From the two parts (rational and irrational) of Eigen values, we have two different pattern of sequence 3, 19, 117… and 1, 6, 37…

The important point is that the terms of these two sequences follow the same recurrence relation.

3. CONCLUSIONS

Above discussion and algebraic operations on matrices and identifying different sequences from the corresponding terms of the product matrix and its exponential terms keeps the track open to perform parallel operations and search for more properties. Any well defined sequences and its generator matrix inherit and hence preserves some basic properties.

4. REFERENCES

1. Hazara, A, K, Matrix Algebra. Calculus and Generalized Inverse (Part-I), Viva books, 2009.
2. Hazara, A, K, Matrix Algebra. Calculus and Generalized Inverse (Part-II), Viva books, 2009.
3. Trivedi, R, A, Bhanotar, S, A, Dr. Jha,P, J. Pythagorean Triplets-views, Analysis, and Classification, IOSR JM, e-ISSN: 2278-5728, p-ISSN: 2319-765X, 2015; 11(2): ppIII54-63.
4. Kadiya, S,S, Patel D.R , Dr. Jha, P,J. Matrix Operation on generator matrices of known sequences and important derivations" International Journal of Applied Research 2016:2 (7);p-ISSN: 2394-7500:pp933-938.
5. Patel D.R., Kadiya, S,S., Patel C.B., Dr.Jha, P, J; "Member Triangles of Fermat Family and Geometric Inter-relationship with Standard Sequences" ISBN.978-2-642-24819-9 Proceedings of the International Conference on Emerging Trends in Scientific Research (ICETR-2015).